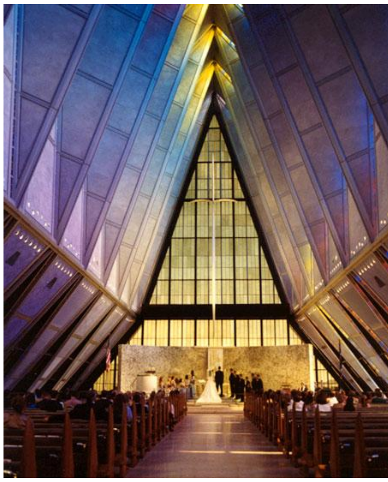
Lesson 4-5: Isosceles & Equilateral Triangles

Isosceles Triangles

Probably one of the most common triangles you see in the world around you is the isosceles triangle. When you look at a roof from the side you are seeing an isosceles triangle: each slope side is the same length. When you look at a bridge, the supports are often isosceles triangles. They are used because they evenly spread the load and are very strong.

Not only are they very strong but they are pleasing to the eye. Because of their symmetry and balance they are used in art and architecture. Here are some pictures of the Air Force Academy's Cadet Chapel that shows isosceles triangles used both for structural strength and beauty.




Characteristics of Isosceles Triangles

Take a piece of paper, construct a large isosceles triangle, and cut it out.

The congruent sides are called the *legs*. The angle at which the congruent sides meet is called the *vertex angle*. The other two angles (at the base of the congruent sides) are called *base angles*. Label the vertex angle A, the others B and C. The side opposite the vertex angle is called the *base*.

Fold the triangle in half from the vertex angle and open it back up. Now, take a minute or so to examine and play with this triangle.

Did you make any interesting observations? Obviously we can see the two congruent sides. Did you notice anything about the base angles? They look congruent don't they?

What about the base? How does it relate to the fold line? The fold line is a perpendicular bisector of the base side!

These conjectures suggest a theorem…

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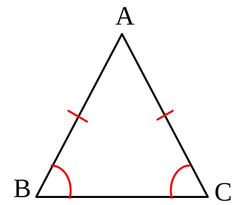
Theorem 4-3 Isosceles Triangle Theorem



If two sides of a triangle are congruent, then the angles opposite those sides are congruent (this is the first part of our conjecture).

Euclid's bridge

In Euclid's Elements, he provided a proof for the Isosceles Triangle Theorem using a diagram creating a figure with a bridge-like appearance. The proof was so complicated and difficult that in medieval times it came to be viewed as the first real test of a student's intelligence and a "bridge" that must crossed to be considered a true student of mathematics. It came to be called the *pons asinorum*, which is Latin for "bridge of asses."

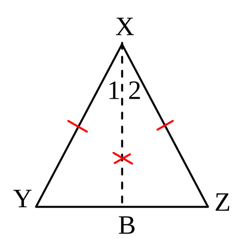
Well thanks to our last lesson we can prove this theorem with a much simpler method. First, let's state this mysterious Isosceles Triangle Theorem.

Proof of Isosceles Triangle Theorem

The trick is to leverage our last lesson: CPCTC. If we have to triangles we can prove are congruent, then all corresponding parts are congruent. So…all we need is two triangles. Looking at the diagram above, we only have one triangle. Look at you cut out isosceles triangle; does that give you an idea? <u>What if we constructed the bisector of the vertex angle to the base?</u> Then we'd have two triangles we could work with! We have the congruent legs, the bisected angle and the shared side.

Proof

Given: $\overline{XY} \cong \overline{XZ}, \overline{XB}$ bisects $\angle YXZ$
Prove: $\angle Y \cong \angle Z$



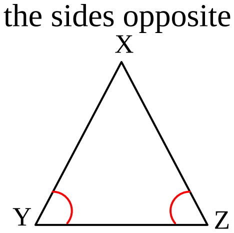
Proof: $\overline{XY} \cong \overline{XZ}$	Given
$\angle 1 \cong \angle 2$	Defn. angle bisector
$\overline{XB} \cong \overline{XB}$	Reflexive POC
$\Delta XYB \cong \Delta XZB$	SAS
$\angle Y \cong \angle Z$	CPCTC
Q.E.D.	

And now, we have crossed the *pons asinorum* and proven ourselves worthy students of mathematics!

Theorem 4-4 Converse of Isosceles Triangle Theorem

If two angles of a triangle are congruent, then the sides opposite the angles are congruent.

$\overline{XY} \cong \overline{XZ}$



Proof of Theorem 4-4

Sorry, you get to do this as an exercise. ☺

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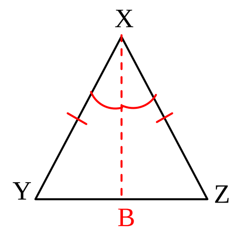
Theorem 4-5

The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base (this is the second part of our conjecture).

$\overline{XB} \perp \overline{YZ}$ and $\overline{XB}$ bisects $\overline{YZ}$

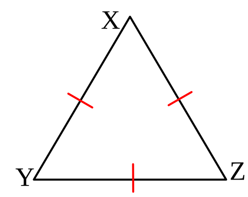


Proof of Theorem 4-5

Sorry, you also get to do this as an exercise! ☺

Corollary to Theorem 4-3

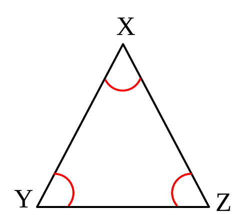
OK, what the heck is a corollary? It is simply a statement that follows immediately from a theorem. You could say it "falls out of" the theorem…just shake the theorem up and out it pops! Well, not literally, but you get the idea…it is basically a "duh!" statement. Here it is:



If a triangle is equilateral, then the triangle is equiangular.

$\overline{XY} \cong \overline{YZ} \cong \overline{XZ} \Rightarrow \angle X \cong \angle Y \cong \angle Z$

If you think about it, it makes sense. Take two of the sides (say $\overline{XY}$ and $\overline{XZ}$). They are congruent (because the triangle is equilateral) and form an isosceles triangle…hence the opposite angles ($\angle Z$ and $\angle Y$) are congruent. Now take the third side $\overline{YZ}$ and one of the original two (say $\overline{XY}$) and do it again. This says that $\angle X$ and $\angle Z$ are congruent. By the transitive POC all three angles are congruent.



Corollary to Theorem 4-4

If a triangle is equiangular, then the triangle is equilateral.

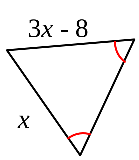
$\angle X \cong \angle Y \cong \angle Z \Rightarrow \overline{XY} \cong \overline{YZ} \cong \overline{XZ}$

You can follow logic similar that for the first corollary to "prove" this.

Examples – not in the book

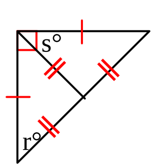
Find the values of the variables.

1)



$x = 3x - 8$ (Isosceles triangle by Conv. of Isos. Δ Thm)
$x = 4$

2)



Using big triangle: $90 + r + r = 180$ ($\Delta\angle$ sum thm & isos Δ base $\angle$'s $\cong$)
$r = 45$
Using small triangle: $s = r$ (both s and r are base $\angle$'s)
$s = 45$

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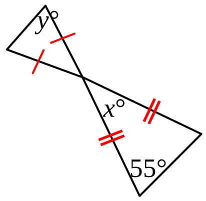
3)



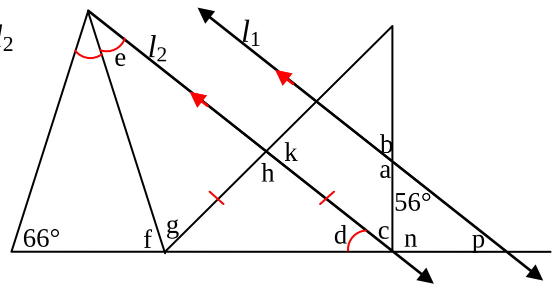
$x + 2 \bullet 55 = 180$ (Isosceles triangle)

$x = 70$

$2 \bullet y + x = 180$ (Vert. $\angle$'s and isosceles triangle)

$y = 55$

4) $l_1 \parallel l_2$



$a = 124; b = 56; c = 56; d = 38; e = 38; f = 76; g = 66; h = 104; k = 76; n = 86$

a: $a = 180 - 56 = 124$ (suppl $\angle$'s)

b: $b = 56$ (vert $\angle$'s)

c: $c = 56$ (alt. int. $\angle$'s)

$d = e$ ($\cong \angle$ markings)

d, e: $d + e + e + 66 = 180$; $3e + 66 = 180$; $e = 38$ ($\Delta \angle$ sum Thm & $\cong \angle$ markings)

f: $e + 66 + f = 180$; $f = 180 - 38 - 66 = 76$ ($\Delta \angle$ sum Thm & $\cong \angle$ markings)

g: $f + g + d = 180$; $76 + g + 38 = 180$; $g = 66$ (Isosceles Δ base $\angle$'s with d)

h: $h + d + d = 180$; $h + 38 + 38 = 180$; $h = 104$ (Isosceles Δ base $\angle$'s with d)

k: $k + h = 180$; $k + 104 = 180$; $k = 76$ (suppl $\angle$'s)

n: $n + c + d = 180$; $n + 56 + 38 = 180$; $n = 86$ ($\Delta \angle$ sum Thm)

Assigned homework

p. 213 #1, 2, 7-18, 21-26, 31, 34-38, 40, 41, 46-50